On the approximation of isochoric motions of fluids under different flow conditions

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Abstract

There has been considerable interest, ever since the development of the approximation by Oberbeck and Boussinesq concerning fluids that are mechanically incompressible but thermally compressible, in giving a rigorous justification for the same. For such fluids, it would be natural to assume that the determinant of the deformation gradient (which is a measure of the volume change of the body) depends on the temperature. However, such an assumption has the attendant drawbacks of the specific heat of the fluid at constant volume being zero and the speed of sound in the fluid being complex. In this paper, we consider a generalization of the Oberbeck-Boussinesq approximation wherein the volume change depends both on the temperature and on the pressure that the fluid is subject to. We show that within the context of this generalization, the specific heat at constant volume can be defined meaningfully, and it is not zero.

1 Introduction

All real bodies are compressible. In fact, if a sufficiently high mean normal compressive stress (mechanical pressure) is employed, the body undergoes a reduction in volume. However, it is possible that some bodies do not undergo a significant reduction in volume over a sufficiently large range of pressures and can hence be approximated as being incompressible in that range. When

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the range of pressures are what is considered 'normal', in view of day to day applications, the body is considered to be incompressible. Of course, what is deemed to be a 'significant reduction in volume' is quite arbitrary and it boils down to whether neglecting the volume reduction and modeling the body as an incompressible body yet captures the essential features of the response of the body when subject to external stimuli. Most liquids can be approximated as incompressible liquids provided the pressures to which they are subject to are not very high. On the other hand, if the range of pressures to which the liquid is subject is very large, then volume changes do take place and, moreover, properties that characterize a fluid like the viscosity may change exponentially with pressure (see Barus [4] and Bair and Koptke [1]). A detailed discussion of the literature pertinent to the variation of viscosity with pressure prior to 1934 can be found in the book by Bridgman [5]. More recent experimental work concerning the variation of viscosity with pressure is discussed in the paper by Rajagopal [20]. In addition to the dependence of the viscosity on the pressure, it is well established that the viscosity also depends on the temperature. The fact that the properties of a fluid can depend on both the pressure and the temperature leads one to very interesting consequences which we will discuss later.

A useful and much used approximation in fluid mechanics is the simplification of the Navier-Stokes equation that was independently introduced by Oberbeck [17, 18] and Boussinesq [3] under the assumption that the fluid is mechanically incompressible but is compressible or expansible due to temperature changes. The approximation is discussed in detail later and is based on the perturbation of the governing equations using an appropriate non-dimensional parameter. The mathematical representation that the volume change depends on the temperature is:

$$\det \mathbf{F} = f(\theta), \tag{1}$$

that is, the determinant of the deformation gradient $\mathbf{F}$ is a function of the absolute temperature θ.

Rayleigh [25], Jeffreys [12] and Chandrasekhar [6] were amongst the early investigators that appealed to the Oberbeck-Boussinesq approximation. There has been considerable effort expended to justify the Oberbeck-Boussinesq approximations but most of these attempts [14, 27, 28, 26, 11] are flawed as pointed out by Rajagopal *et al.* [21]. Rajagopal *et al.* [21] used the constraint (1) and carried out a perturbation analysis to show that the Oberbeck-Boussinesq approximation is not an approximation that gathers terms that are all of the same order in the perturbation parameter. This study of Rajagopal *et al.* [21] has been extended by Rajagopal *et al.* [22] for fluids with pressure dependent viscosity.

Müller [16] defines a body to be incompressible if the density and the internal energy depend only on the temperature and do not depend on the pressure (mean normal stress). Appealing to the entropy principle Müller [16] shows that such a fluid cannot undergo changes in volume due to temperature changes. Such behavior is clearly contradicted by experiments which show that volume changes do take place with temperature. Motivated by the fact that experimental evidence clearly contradicts Müller's conclusion, Gouin *et al.* [8] studied bodies whose density depends only on the temperature (they did not also require that the internal energy depends only on the temperature) and they referred to such bodies as quasi-thermally compressible bodies. We notice that the assumption of Gouin *et al.* [8] is the same as that of (1). The terminology 'quasi' according to the Oxford English Dictionary means "resembling or simulating, but not really the same as" and such a terminology is not an appropriate description of (1). In fact, equation (1) clearly implies that the body can change its volume with temperature and if the temperature remains a constant no volume change can take place. As the body clearly changes volume as the temperature changes, it is thermally compressible, there is nothing which requires one to add the terminology 'quasi' in characterizing its thermal compressibility.

The analysis of Gouin *et al.* [8], as well as the earlier analysis of Rajagopal *et al.* [21] imply that the specific heat of the body under consideration at constant volume is zero. This is definitely not in keeping with physical observation. In fact, Gouin *et al.* [8] devote sub-sections of their paper to a discussion of the response of water and of Gum rubber, the specific heat at constant volume of both is non-zero. Of course, one recognizes that in making idealizations or approximations in order to capture certain aspects of a physical phenomenon one may miss describing some other phenomenon that is not of primary interest. That being said, it would be preferable to be in keeping with physics as much as possible and in this paper we investigate whether it would be possible to develop a theory which does not lead to paradoxical results (such as the specific heat at constant volume being zero) by generalizing (1).

As we shall show, assumption (1) leads to two more physically unrealistic deductions: thermodynamic instability (the specific entropy fails to be a concave function of the pressure and specific volume) and imaginary speed of sound. To overcome these drawbacks, Gouin and Ruggeri [9] defined an *extended-quasi-thermal- incompressible fluid* (EQTI), a fluid whose internal energy and specific volume are of the form

$$e(p,\theta) = e_0(\theta) + O(\varepsilon^2), \quad v(p,\theta) = v_0 + \varepsilon v_1(\theta) + \varepsilon^2 v_2(p,\theta), \tag{2}$$

respectively, where v_0 is a constant and $\varepsilon > 0$ is a small parameter that we

shall introduce in section 4. In an attempt to improve Müller's model of incompressibility [16], Gouin and Ruggeri [9] provide a critical threshold for the pressure below which an EQTI fluid can be regarded as incompressible. Here we shall not introduce *ad hoc* assumptions on the form of the constitutive functions (such as (2)) to determine when the idealization of incompressibility is reasonable, instead we shall determine some flow regimes in which a real fluid undergoes exclusively isochoric motions.

In this paper we are interested in developing several approximations for different classes of flows of a fluid with pressure and temperature dependent material properties. For such materials it would be reasonable to modify the assumption (1) by postulating

$$\det \mathbf{F} = f(p, \theta), \tag{3}$$

where

$$p \equiv -\frac{1}{3}\mathrm{tr}\mathbf{T}, \tag{4}$$

is the mean value of the normal stresses of the Cauchy stress tensor $\mathbf{T}$, which is usually referred to as the mechanical pressure [19]. By making the assumption (3), we shall be able to show that the body in question has a meaningful specific heat at constant volume. The earlier work of Rajagopal *et al.* [22] related to fluids with pressure dependent viscosity assumes (1) rather than (3), but, as it will be shown in section 6, assumption (1) leads to the result that the specific heat at constant volume is zero.

The organization of this paper is as follows. In the next section we document the system of equations: the balance of mass, the balance of linear momentum, the balance of energy and the second law of Thermodynamics in the form of the Clausius-Duhem inequality. We introduce a fluid whose properties depend on the mean value of the normal stresses and temperature and investigate the consequences of the governing equations. In section 3 we establish certain thermodynamic identities and follow up in the next section with the non-dimensionalization of the governing equations and the development of a perturbation procedure. We then develop approximate equations that are valid in different flow regimes, one of them being the Oberbeck-Boussinesq approximation. All the approximations in section 4 are derived under the assumption that Reynolds' number is of order of unity or smaller. However, approximations valid at higher Reynolds' numbers (such as those which have recently been employed to study stratified flows [10, 31] turbulence in forced convection [32] and heat transfer in turbulent mixed convection [33]) can be derived from the perturbation analysis developed in section 4. Finally, in section 5 we discuss two more drawbacks arising from the as-

sumption (1): thermodynamic instability and imaginary speed of sound in the fluid.

2 Governing equations

We record below the balance of mass, linear momentum and energy, as well as the second law of Thermodynamics (in the form of the Clausius-Duhem inequality):

$$\dot{\rho} + \rho \mathrm{div}\mathbf{v} = 0, \tag{5}$$

$$\rho\dot{\mathbf{v}} = \mathrm{div}\mathbf{T} + \rho\mathbf{b}, \tag{6}$$

$$\rho\dot{e} + \mathrm{div}\mathbf{q} = \mathbf{T}\cdot\mathbf{D} + \rho r, \tag{7}$$

$$\rho\dot{\eta} \geq \rho\frac{r}{\theta} - \mathrm{div}\left(\frac{\mathbf{q}}{\theta}\right), \tag{8}$$

where ρ is the density, $\mathbf{v}$ the velocity field, $\mathbf{T}$ the Cauchy stress tensor, $\mathbf{b}$ the specific external body force field, e the specific internal energy, r the specific radiant heating, θ the temperature, $\mathbf{q}$ the heat flux vector, $\mathbf{D}$ the symmetric velocity gradient and η the specific entropy. We are assuming that the stress is symmetric and that the balance of angular momentum is met (we are assuming that there are no body couples).

We shall allow the fluid to be slightly compressible due to variations in the pressure and temperature and hence assume that (3) holds.

We assume that fluid motion is sufficiently smooth so that the derivatives that are taken are meaningful. Next, we differentiate the determinant of the deformation gradient with respect to time and obtain that

$$\mathrm{div}\mathbf{v} = -k_T(p,\theta)\dot{p} + \alpha(p,\theta)\dot{\theta}, \tag{9}$$

where

$$k_T = -\frac{1}{f}\frac{\partial f}{\partial p}, \quad \alpha = \frac{1}{f}\frac{\partial f}{\partial \theta} \tag{10}$$

are the isothermal compressibility and the coefficient of thermal expansion, respectively, that are related through the integrability condition

$$\frac{\partial k_T}{\partial \theta} = -\frac{\partial \alpha}{\partial p}. \tag{11}$$

From (5) and (9) we deduce that

$$\frac{\dot{\rho}}{\rho} = k_T\dot{p} - \alpha\dot{\theta}. \tag{12}$$

Hence

$$\mathrm{d}\rho = \rho\left(k_T \mathrm{d}p - \alpha \mathrm{d}\theta\right), \tag{13}$$

and, denoting $v = 1/\rho$ the specific volume,

$$\mathrm{d}v = -\frac{k_T}{\rho}\mathrm{d}p + \frac{\alpha}{\rho}\mathrm{d}\theta. \tag{14}$$

Next, we introduce the enthalpy

$$h = e + pv \tag{15}$$

and the Gibbs free enthalpy

$$g = h - \theta\eta, \tag{16}$$

and combine these two thermodynamic potentials with the balance equations (5)–(7) and the Clausius-Duhem inequality to get

$$\rho(\dot{g} + \eta\dot{\theta}) - \dot{p} - p\mathrm{div}\mathbf{v} - \mathbf{T}\cdot\mathbf{D} + \frac{\mathbf{q}}{\theta}\cdot\nabla\theta \leq 0. \tag{17}$$

In Continuum Thermodynamics, the Clausius-Duhem inequality is used to obtain restrictions on constitutive equations, but this procedure is not without its shortcomings which we shall discuss later. From now on we shall regard the mechanical pressure and the temperature as independent variables on which the material parameters of the fluid depend, and, since we are interested in fluids of grade 1 (see Truesdell and Noll [29]), the requirement of material frame indifference and the representation theorems for isotropic functions lead us to consider the following constitutive functions

$$e = \hat{e}(p, \theta, \mathrm{tr}\mathbf{D}), \tag{18}$$

$$\eta = \hat{\eta}(p, \theta, \mathrm{tr}\mathbf{D}), \tag{19}$$

$$\mathbf{T} = -p\mathbf{I} + 2\mu(p,\theta)\left[\mathbf{D} - \frac{1}{3}(\mathrm{tr}\mathbf{D})\mathbf{I}\right] \tag{20}$$

and

$$\mathbf{q} = -k(p,\theta)\nabla\theta, \tag{21}$$

where μ is the (shear) viscosity of the fluid and k is the heat conductivity. Finally, we introduce the specific heats at constant pressure and at constant volume [7] through

$$c_p = \left(\frac{\partial h}{\partial\theta}\right)_p, \quad c_v = \left(\frac{\partial e}{\partial\theta}\right)_v, \tag{22}$$

respectively, and the specific heat ratio $\gamma = c_p/c_v$.

Inserting (20), (21) and (9) into (17) yields the inequality

$$\left(\rho\frac{\partial g}{\partial p} - 1\right)\dot{p} + \rho\left(\frac{\partial g}{\partial \theta} + \eta\right)\dot{\theta} + \rho\frac{\partial g}{\partial\,\mathrm{tr}\mathbf{D}}\overline{\mathrm{tr}\mathbf{D}}^{\cdot} \\ -2\mu\left[\|\mathbf{D}\|^2 - \frac{1}{3}(\mathrm{tr}\mathbf{D})^2\right] - \frac{k}{\theta}\|\nabla\theta\|^2 \leq 0, \tag{23}$$

that holds true for any thermodynamical processes, *i.e.*, for any fields ρ, $\mathbf{v}$ and θ satisfying the balance equations (5)–(7). Therefore, by using standard arguments in Continuum Thermodynamics, we deduce that

$$\frac{\partial g}{\partial p} = \frac{1}{\rho}, \quad \frac{\partial g}{\partial \theta} = -\eta, \quad \frac{\partial g}{\partial\,\mathrm{tr}\mathbf{D}} = 0 \tag{24}$$

and that the constitutive functions for the fluid viscosity μ and the thermal conductivity k are non-negative.

A few remarks concerning the use of the Clausius-Duhem inequality to obtain restrictions on the constitutive relations are in order. In employing the Clausius-Duhem inequality to obtain restrictions on the forms of the constitutive relation one subjects the body to arbitrary processes. It is natural to enquire whether such a procedure is reasonable. Let us consider a piece of rubber that is being deformed. We might suppose that the stress in such a material depends only on the deformation gradient. But in making such an assumption there is the tacit understanding that the piece of rubber is not subject to arbitrary processes, for if we were to subject the piece of rubber to a very high strain rate it would crystallize. Thus, in fact, if we want to be true to physics, the constitutive relation ought to say that the stress depends on the deformation gradient provided that the strain rate is not larger than some prescribed value, and this may change from one piece of rubber to another. Thus, instead of using the Clausius-Duhem inequality to obtain both necessary and sufficient conditions so that the rate of entropy production is non-negative, we can merely make appropriate assumptions that are sufficient to guarantee that the rate of entropy production is non-negative. For the body under consideration, the conditions (24) are sufficient to assure that the rate of entropy production is non-negative.

From (24) we deduce that the differential of the Gibbs free enthalpy is

$$\mathrm{d}g = \frac{1}{\rho}\mathrm{d}p - \eta\mathrm{d}\theta \tag{25}$$

and the constitutive functions for the specific internal energy and entropy are, respectively, of the form $e = \hat{e}(p,\theta)$, and $\eta = \hat{\eta}(p,\theta)$ (namely, both the

specific internal energy and the specific entropy do *not* depend on tr$\mathbf{D}$), with

$$\left(\frac{\partial\hat{\eta}}{\partial p}\right)_{\theta} = -\frac{\alpha}{\rho}. \tag{26}$$

Next, combining (15) and (16) with (25) yields

$$\theta \mathrm{d}\eta = \mathrm{d}e + p\mathrm{d}v \tag{27}$$

which in turn leads to

$$\theta\left(\frac{\partial\hat{\eta}}{\partial\theta}\right)_{v} = \left(\frac{\partial\hat{e}}{\partial\theta}\right)_{v} = c_v, \tag{28}$$

and then by virtue of (13) and (14) we obtain

$$\theta\left(\frac{\partial\hat{\eta}}{\partial\theta}\right)_{p} = \left(\frac{\partial\hat{e}}{\partial\theta}\right)_{p} + \frac{\alpha}{\rho}p = \left(\frac{\partial h}{\partial\theta}\right)_{p} = c_p. \tag{29}$$

As far as the specific internal energy is concerned, from $(24)_1$, (26) and (29) we have

$$\left(\frac{\partial\hat{e}}{\partial p}\right)_{\theta} = \frac{k_T p - \alpha\theta}{\rho}, \quad \left(\frac{\partial\hat{e}}{\partial\theta}\right)_{p} = c_p - \frac{\alpha}{\rho}p. \tag{30}$$

Finally, by using (20), (9) and (30) the equations of balance of linear momentum (6) and energy (7) can be expressed as

$$\rho\dot{\mathbf{v}} = -\nabla p + 2\mathrm{div}\left\{\mu\left[\mathbf{D} - \frac{1}{3}(\mathrm{div}\mathbf{v})\mathbf{I}\right]\right\} + \rho\mathbf{b} \tag{31}$$

and

$$-\alpha\theta\dot{p} + \rho c_p\dot{\theta} = \mathrm{div}(k\nabla\theta) + 2\mu\left[\|\mathbf{D}\|^2 - \frac{1}{3}(\mathrm{div}\mathbf{v})^2\right] + \rho r, \tag{32}$$

respectively. Equations (9), (12), (31) and (32) constitute a system of partial differential equations for determining the thermodynamic fields ρ, $\mathbf{v}$, p and θ.

3 Thermodynamic identities

In this section we derive some thermodynamic identities that are fundamental to determining the most appropriate non-dimensionalization for the problem under consideration. While there is some overlap between the results established in this section and that of section 2 of [9], the results concerning

the specific heat at constant volume c_v are being recorded for the first time as there is no discussion of the conditions guaranteeing the thermodynamic stability that involves the specific heat at constant volume in [9]. More importantly, we show that the assumption (1) leads to the physically unrealistic deduction that $c_v = 0$. For the sake of completeness and clarity we keep the discussion concerning k_T and c_p in the section.

We start by noticing that, in virtue of (26) and (29), the differential of the specific entropy is

$$\mathrm{d}\eta = -\frac{\alpha}{\rho}\mathrm{d}p + \frac{c_p}{\theta}\mathrm{d}\theta. \tag{33}$$

Combining (33) with (14) yields

$$\mathrm{d}\eta = \left(\frac{c_p}{\theta} - \frac{\alpha^2}{\rho k_T}\right)\mathrm{d}\theta + \frac{\alpha}{k_T}\mathrm{d}v \tag{34}$$

and

$$\mathrm{d}\rho = \left(\rho k_T - \frac{\alpha^2\theta}{c_p}\right)\mathrm{d}p - \frac{\rho\alpha\theta}{c_p}\mathrm{d}\eta. \tag{35}$$

Then, from (28) and (34) it follows that the specific heats at constant volume and at constant pressure are related through

$$c_v = \theta\left(\frac{\partial\eta}{\partial\theta}\right)_v = c_p - \frac{\alpha^2\theta}{\rho k_T}. \tag{36}$$

In order to ensure thermodynamic stability, the specific entropy must be a concave function of the specific internal energy and the specific volume, that is, in virtue of (27), the hessian matrix of the specific entropy

$$H_\eta(e,v) = \begin{pmatrix} -\dfrac{1}{\theta^2}\left(\dfrac{\partial\theta}{\partial e}\right)_v & -\dfrac{1}{\theta^2}\left(\dfrac{\partial\theta}{\partial v}\right)_e \\ \left(\dfrac{\partial(p/\theta)}{\partial e}\right)_v & \left(\dfrac{\partial(p/\theta)}{\partial v}\right)_e \end{pmatrix} \tag{37}$$

must be negative semi-definite. To find conditions guaranteeing the negative semi-definiteness of the matrix (37), from (13), (14) and (30) we deduce that

$$\mathrm{d}\theta = \frac{1}{c_v}\mathrm{d}e + \frac{k_T p - \alpha\theta}{k_T c_v}\mathrm{d}v \tag{38}$$

and

$$\mathrm{d}p = \frac{\alpha}{k_T c_v}\mathrm{d}e + \frac{p\alpha - \rho c_p}{k_T c_v}\mathrm{d}v. \tag{39}$$

In virtue of (38) and (39), the hessian matrix (37) becomes

$$H_\eta(e,v) = \begin{pmatrix} -\dfrac{1}{\theta^2 c_v} & \dfrac{\alpha\theta - k_T p}{\theta^2 k_T c_v} \\ \dfrac{\alpha\theta - k_T p}{\theta^2 k_T c_v} & \dfrac{2p\alpha\theta - \rho c_p \theta - k_T p^2}{\theta^2 k_T c_v} \end{pmatrix} \tag{40}$$

that, by means of Sylvester's criterion, is negative semi-definite if and only if

$$H_{\eta_{11}} = -\frac{1}{\theta^2 c_v} \leq 0 \quad \text{and} \quad \det H_\eta = \frac{\rho}{k_T \theta c_v} \geq 0. \tag{41}$$

Hence,

$$c_v > 0 \quad \text{and} \quad k_T > 0 \tag{42}$$

are necessary and sufficient conditions for thermodynamic stability.

The thermodynamic stability conditions (42) imply other conditions. From (36) and (42) we deduce that the specific heat at constant pressure c_p is positive as

$$c_p > \frac{\alpha^2 \theta}{\rho k_T}, \tag{43}$$

in agreement with the stability condition in [9], and the specific heat ratio $\gamma > 1$. Then, in virtue of (36) we deduce that the isothermal compressibility and the coefficient of thermal expansion are related through

$$\alpha^2 = c_v(\gamma - 1)\frac{\rho}{\theta} k_T. \tag{44}$$

Moreover, denoting c the speed of sound in the fluid, from (35) and (36) we deduce that

$$\frac{1}{c^2} \equiv \left(\frac{\partial \rho}{\partial p}\right)_\eta = \rho k_T - \frac{\alpha^2 \theta}{c_p} = \frac{\rho k_T}{\gamma}. \tag{45}$$

Hence, the isothermal compressibility may be determined through the relation

$$k_T = \frac{\gamma}{\rho c^2}. \tag{46}$$

Finally, combining identities (44) and (46) yields

$$\alpha^2 = \frac{c_p(\gamma - 1)}{c^2 \theta}. \tag{47}$$

4 Approximations

Let us consider a horizontal fluid layer Ω of thickness d and assume that gravity is the only body force acting on the fluid. In other words, we assume that $\mathbf{b} = -g\mathbf{k}$, where g is the acceleration due to gravity and $\mathbf{k}$ is the direction in which the fluid layer is bounded. We assume also that no heat is supplied, *i.e.*, $r = 0$. In order to non-dimensionalize the equations governing the fluid motion, we choose a convenient reference state $(p_{\text{ref}}, \theta_{\text{ref}})$ and introduce the following scales

$$\begin{gathered}
\mathbf{x}^* = \frac{\mathbf{x}}{d}, \quad \mathbf{v}^* = \frac{\mathbf{v}}{V}, \quad \rho^* = \frac{\rho}{\rho_{\text{ref}}}, \quad t^* = \frac{V}{d}t, \\
p^* = \frac{p - p_{\text{ref}}}{\rho_{\text{ref}} g d}, \quad \theta^* = \frac{\theta - \theta_{\text{ref}}}{\theta_M - \theta_m}, \quad \alpha^* = \frac{\alpha}{\alpha_{\text{ref}}}, \quad c^* = \frac{c}{c_{\text{ref}}}, \\
\mu^* = \frac{\mu}{\mu_{\text{ref}}}, \quad c_p^* = \frac{c_p}{c_{p_{\text{ref}}}}, \quad k_T^* = \frac{\rho_{\text{ref}} c_{\text{ref}}^2}{\gamma_{\text{ref}}} k_T.
\end{gathered} \tag{48}$$

In (48) the subscript 'ref' indicates that the corresponding material parameters are evaluated at the reference state $(p_{\text{ref}}, \theta_{\text{ref}})$, V is the reference velocity, $\theta_M = \max_\Omega \theta$ and $\theta_m = \min_\Omega \theta$. Hereinafter, we choose θ_M as the reference temperature.

Substituting (48) into (9), (12), (31) and (32) yields the dimensionless equations (omitting the asterisks for convenience)

$$\text{div}\mathbf{v} = -\gamma_{\text{ref}} \frac{\text{Ma}^2}{\text{Fr}^2} k_T \dot{p} + \alpha_{\text{ref}} (\theta_M - \theta_m) \alpha \dot{\theta}, \tag{49}$$

$$\dot{\rho} = \gamma_{\text{ref}} \frac{\text{Ma}^2}{\text{Fr}^2} \rho k_T \dot{p} - \alpha_{\text{ref}} (\theta_M - \theta_m) \rho \alpha \dot{\theta}, \tag{50}$$

$$\text{Fr}^2 \rho \dot{\mathbf{v}} = -\nabla p + 2\frac{\text{Fr}^2}{\text{Re}} \text{div} \left\{ \mu \left[\mathbf{D} - \frac{1}{3} (\text{div}\mathbf{v}) \mathbf{I} \right] \right\} - \rho \mathbf{k}, \tag{51}$$

$$\begin{aligned}
-\alpha_{\text{ref}} (\theta_M - \theta_m) \frac{\text{ReBr}}{\text{Fr}^2} \alpha \left(\theta + \frac{1}{\text{Ca}} \right) \dot{p} + \text{Pe} \rho c_p \dot{\theta} = \text{div}(k \nabla \theta) \\
+ 2\text{Br} \mu \left[\|\mathbf{D}\|^2 - \frac{1}{3} (\text{div}\mathbf{v})^2 \right],
\end{aligned} \tag{52}$$

where

$$\begin{gathered}
\text{Ma} = \frac{V}{c_{\text{ref}}}, \quad \text{Fr}^2 = \frac{V^2}{gd}, \quad \text{Re} = \frac{\rho_{\text{ref}} V d}{\mu_{\text{ref}}}, \\
\text{Br} = \frac{\mu_{\text{ref}} V^2}{k_{\text{ref}} (\theta_M - \theta_m)}, \quad \text{Pe} = \frac{\rho_{\text{ref}} c_{p_{\text{ref}}} V d}{k_{\text{ref}}}, \quad \text{Ca} = \frac{\theta_M - \theta_m}{\theta_M}
\end{gathered} \tag{53}$$

are the Mach, second Froude, Reynolds, Brinkman, Péclet and Carnot numbers, respectively.

Hereinafter, we assume that the material parameters α, k_T, c_p and k are analytic functions and limit our analysis to the departures of the pressure and temperature from the reference state $(p_{\mathrm{ref}}, \theta_M)$ for which we can write

$$\alpha(p,\theta) = \sum_{j_1,j_2=0}^{+\infty} \frac{1}{j_1!j_2!} \frac{\partial^{j_1+j_2}\alpha}{\partial p^{j_1}\partial\theta^{j_2}}(0,0) p^{j_1}\theta^{j_2}, \tag{54}$$

$$k_T(p,\theta) = \sum_{j_1,j_2=0}^{+\infty} \frac{1}{j_1!j_2!} \frac{\partial^{j_1+j_2}k_T}{\partial p^{j_1}\partial\theta^{j_2}}(0,0) p^{j_1}\theta^{j_2}, \tag{55}$$

$$c_p(p,\theta) = \sum_{j_1,j_2=0}^{+\infty} \frac{1}{j_1!j_2!} \frac{\partial^{j_1+j_2}c_p}{\partial p^{j_1}\partial\theta^{j_2}}(0,0) p^{j_1}\theta^{j_2} \tag{56}$$

and

$$k(p,\theta) = \sum_{j_1,j_2=0}^{+\infty} \frac{1}{j_1!j_2!} \frac{\partial^{j_1+j_2}k}{\partial p^{j_1}\partial\theta^{j_2}}(0,0) p^{j_1}\theta^{j_2}. \tag{57}$$

From the integrability conditions (11), the expansions (54) and (55) and the scales (48) we deduce that

$$\gamma_{\mathrm{ref}} \frac{\mathrm{Ma}^2}{\mathrm{Fr}^2} \frac{\partial^{j_1+j_2}k_T}{\partial p^{j_1-1}\partial\theta^{j_2+1}}(0,0) = -\alpha_{\mathrm{ref}}(\theta_M - \theta_m) \frac{\partial^{j_1+j_2}\alpha}{\partial p^{j_1}\partial\theta^{j_2}}(0,0), \tag{58}$$

for all $(j_1, j_2) \in \mathbb{N} \times \mathbb{N}_0$. In virtue of the identities (58) we can integrate equation (50) to obtain

$$\begin{aligned} \rho = \exp\Bigg[&- \alpha_{\mathrm{ref}}(\theta_M - \theta_m) \sum_{j_1,j_2=0}^{+\infty} \frac{1}{j_1!(j_2+1)!} \frac{\partial^{j_1+j_2}\alpha}{\partial p^{j_1}\partial\theta^{j_2}}(0,0) p^{j_1}\theta^{j_2+1} \\ &+ \gamma_{\mathrm{ref}} \frac{\mathrm{Ma}^2}{\mathrm{Fr}^2} \sum_{j_1=0}^{+\infty} \frac{1}{(j_1+1)!} \frac{\partial^{j_1}k_T}{\partial p^{j_1}}(0,0) p^{j_1+1} \Bigg]. \end{aligned} \tag{59}$$

Evaluating the identity (47) at the reference state $(p_{\mathrm{ref}}, \theta_M)$ yields the following relation

$$[\alpha_{\mathrm{ref}}(\theta_M - \theta_m)]^2 = \frac{\mathrm{Ma}^2}{\mathrm{E}} \mathrm{Ca}(\gamma_{\mathrm{ref}} - 1), \tag{60}$$

where

$$\mathrm{E} = \frac{V^2}{c_{p_{\mathrm{ref}}}(\theta_M - \theta_m)} \tag{61}$$

is the Eckert number. We henceforth consider thermodynamic processes such that

$$O\left(\frac{\mathrm{E}}{\mathrm{Fr}^2}\right) \leq O(\mathrm{Ca}(\gamma_{\mathrm{ref}}-1)), \quad \alpha_{\mathrm{ref}}(\theta_M-\theta_m) \ll 1. \tag{62}$$

Then, in virtue of (60) and $(62)_1$, $\mathrm{Ma}^2/\mathrm{Fr}^2$ is of the same order as $\alpha_{\mathrm{ref}}^2(\theta_M-\theta_m)^2$ or smaller.

We are now in position to carry out a perturbation analysis with respect to the small parameter $\varepsilon := \alpha_{\mathrm{ref}}(\theta_M-\theta_m)$. Let

$$\mathbf{v} = \sum_{n=0}^{+\infty} \varepsilon^n \mathbf{v}_n, \quad p = \sum_{n=0}^{+\infty} \varepsilon^n p_n, \quad \theta = \sum_{n=0}^{+\infty} \varepsilon^n \theta_n \tag{63}$$

be the power series in ε of the thermodynamic fields $\mathbf{v}$, p and θ. As far as the power series expansion of the fluid density is concerned, it may be derived from (59) and (63) by taking into account the fact that $\gamma_{\mathrm{ref}}\mathrm{Ma}^2/\mathrm{Fr}^2$ is of order $O(\varepsilon^2)$. However, the expression is quite complicated and of no practical interest to our analysis. In our analytical scheme it suffices to know that

$$\begin{aligned}\rho &= 1-\varepsilon \sum_{j_1,j_2=0}^{+\infty} \frac{1}{j_1!(j_2+1)!}\frac{\partial^{j_1+j_2}\alpha}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)p_0^{j_1}\theta_0^{j_2+1} + o(\varepsilon) \\ &= 1-\varepsilon\int_0^{\theta_0} \alpha(p_0,\theta_0)\mathrm{d}\theta_0 + o(\varepsilon),\end{aligned} \tag{64}$$

where $o(\varepsilon)$ accounts for terms of order $O(\varepsilon^2)$ and higher. Therefore, by inserting (63) and (64) into (49), (51) and (52) we deduce a system of equations from which we may derive different approximations:

$$\begin{aligned}\sum_{n=0}^{+\infty}\varepsilon^n \mathrm{div}\mathbf{v}_n = &-\gamma_{\mathrm{ref}}\frac{\mathrm{Ma}^2}{\mathrm{Fr}^2}\sum_{n=0}^{+\infty}\varepsilon^n\left[k_T(p,\theta)\left(\frac{\partial p}{\partial t}+\mathbf{v}\cdot\nabla p\right)\right]_n \\ &+\sum_{n=0}^{+\infty}\varepsilon^{n+1}\left[\alpha(p,\theta)\left(\frac{\partial\theta}{\partial t}+\mathbf{v}\cdot\nabla\theta\right)\right]_n,\end{aligned} \tag{65}$$

$$\begin{aligned}\mathrm{Fr}^2\sum_{n=0}^{+\infty}\varepsilon^n\left\{\rho\left[\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}\right]\right\}_n &= -\sum_{n=0}^{+\infty}\varepsilon^n\nabla p_n \\ &+2\frac{\mathrm{Fr}^2}{\mathrm{Re}}\sum_{n=0}^{+\infty}\varepsilon^n\left\{\mathrm{div}\left[\mu(p,\theta)\left(\mathbf{D}-\frac{1}{3}(\mathrm{div}\mathbf{v})\mathbf{I}\right)\right]\right\}_n \\ &-\left[1-\varepsilon\int_0^{\theta_0}\alpha(p_0,\theta_0)\mathrm{d}\theta_0+o(\varepsilon)\right]\mathbf{k}\end{aligned} \tag{66}$$

and

$$
\begin{aligned}
&-\frac{\mathrm{ReBr}}{\mathrm{Fr}^2}\sum_{n=0}^{+\infty}\varepsilon^{n+1}\left[\alpha(p,\theta)\left(\theta+\frac{1}{\mathrm{Ca}}\right)\left(\frac{\partial p}{\partial t}+\mathbf{v}\cdot\nabla p\right)\right]_n \\
&+\mathrm{Pe}\sum_{n=0}^{+\infty}\varepsilon^{n}\left[\rho c_p(p,\theta)\left(\frac{\partial\theta}{\partial t}+\mathbf{v}\cdot\nabla\theta\right)\right]_n=\sum_{n=0}^{+\infty}\varepsilon^{n}\left\{\mathrm{div}[k(p,\theta)\nabla\theta]\right\}_n \\
&+2\mathrm{Br}\sum_{n=0}^{+\infty}\varepsilon^{n}\left\{\mu(p,\theta)\left[\|\mathbf{D}\|^2-\frac{1}{3}(\mathrm{div}\mathbf{v})^2\right]\right\}_n .
\end{aligned}
\tag{67}
$$

Since $\gamma_{\mathrm{ref}}\mathrm{Ma}^2/\mathrm{Fr}^2$ is of order $O(\varepsilon^2)$ or smaller, collecting terms of order $O(1)$ in equation (65) leads to

$$
\mathrm{div}\mathbf{v}_0=0, \tag{68}
$$

whence the fluid motions can be regarded as isochoric to a first approximation.

According to the magnitude of the dimensionless numbers occurring in (66) and (67), we may derive different sets of approximate equations.

4.1 Oberbeck-Boussinesq approximation

If the second Froude number is of order $O(\varepsilon)$, the Reynolds number of order of unity, the Brinkman number of order $O(\varepsilon)$ or smaller and the Péclet number of order of unity or greater, then at the leading order equations (66) and (67) are

$$
0=-\nabla p_0-\mathbf{k} \tag{69}
$$

and

$$
\mathrm{Pe}c_p(p_0,\theta_0)\dot{\theta}_0=\mathrm{div}[k(p_0,\theta_0)\nabla\theta_0]. \tag{70}
$$

Obviously, equation (69) may be integrated and, taking the atmospheric pressure as the reference pressure, we can deduce that p_0 coincides with the hydrostatic pressure $p_{\mathrm{h}}=1-z$.

We now notice that equations (68) and (70) are not sufficient to determine all the thermodynamic fields at $O(1)$. Therefore, in order to attain the closure, we collect the terms of order $O(\varepsilon)$ in (66) and, in view of (68), get

$$
\mathrm{Fr}^2\dot{\mathbf{v}}_0=-\varepsilon\nabla p_1+2\frac{\mathrm{Fr}^2}{\mathrm{Re}}\mathrm{div}[\mu(p_{\mathrm{h}},\theta_0)\mathbf{D}_0]+\varepsilon\int_0^{\theta_0}\alpha(p_{\mathrm{h}},\theta_0)\mathrm{d}\theta_0\mathbf{k}. \tag{71}
$$

Now equations (68), (70) and (71) form a closed system, in which p_1 represents the hydrodynamic pressure. Finally, summing term by term (69) and

(71), renaming $p = p_0 + \varepsilon p_1$ and re-dimensionalizing the resulting equation, (68) and (70) yield the Oberbeck-Boussinesq approximation

$$\begin{cases} \rho_{\rm ref}\dot{\mathbf{v}} = -\nabla p + 2{\rm div}[\mu(p_{\rm h},\theta)\mathbf{D}] - \rho_{\rm ref} g\left[1 - \displaystyle\int_{\theta_{\rm ref}}^{\theta} \alpha(p_{\rm h},\theta){\rm d}\theta\right]\mathbf{k}, \\ {\rm div}\mathbf{v} = 0, \\ \rho_{\rm ref} c_p \dot{\theta} = {\rm div}[k(p_{\rm h},\theta)\nabla\theta], \end{cases} \tag{72}$$

where the dimensionalized hydrostatic pressure is given by the well-known Stevin's law $p_{\rm h} = \rho_{\rm ref} g(d - z)$.

For highly viscous fluids the Brinkman number might not be small. Hence, if the Brinkman number is of order of unity or greater, from (67) we deduce that frictional heating has to be included into equation $(72)_3$ and obtain

$$\rho_{\rm ref} c_p \dot{\theta} = {\rm div}[k(p_{\rm h},\theta)\nabla\theta] + 2\mu(p_{\rm h},\theta)\|\mathbf{D}\|^2. \tag{73}$$

4.2 Stokes-Oberbeck-Boussinesq approximation

If the second Froude number is of order $O(\varepsilon^2)$ or smaller, the ratio ${\rm Fr}^2/{\rm Re}$ of order $O(\varepsilon)$, the Brinkman numer of order $O(\varepsilon)$ or smaller and the Péclet number of order of unity or greater, then we get a modified Oberbeck-Boussinesq approximation that generalizes the Stokes approximation to the non-isothermal case: the so-called *Stokes-Oberbeck-Boussinesq* approximation [30]. By following the same procedure as before the Stokes-Oberbeck-Boussinesq approximation is found to be

$$\begin{cases} \mathbf{0} = -\nabla p + 2{\rm div}[\mu(p_{\rm h},\theta)\mathbf{D}] - \rho_{\rm ref} g\left[1 - \displaystyle\int_{\theta_{\rm ref}}^{\theta} \alpha(p_{\rm h},\theta){\rm d}\theta\right]\mathbf{k}, \\ {\rm div}\mathbf{v} = 0, \\ \rho_{\rm ref} c_p \dot{\theta} = {\rm div}[k(p_{\rm h},\theta)\nabla\theta]. \end{cases} \tag{74}$$

4.3 Non-isothermal Stokes approximation

Assume that $O({\rm Fr}^2) = O({\rm Re}) = O(\varepsilon^2)$ and the Péclet and Brinkman numbers are of order of unity or greater. Then collecting the terms of order $O(1)$ in equations (66) and (67) and re-dimensionalizing lead to the non-isothermal Stokes equations:

$$\begin{cases} \mathbf{0} = -\nabla p + 2{\rm div}[\mu(p,\theta)\mathbf{D}] - \rho_{\rm ref} g\mathbf{k}, \\ {\rm div}\mathbf{v} = 0, \\ \rho_{\rm ref} c_p \dot{\theta} = {\rm div}[k(p,\theta)\nabla\theta] + 2\mu(p,\theta)\|\mathbf{D}\|^2. \end{cases} \tag{75}$$

Equations (75) may be used to study the non-isothermal creeping flows in highly viscous fluids (see, for instance, [15] and the references therein).

4.4 Non-isothermal Navier-Stokes equations for a pressure- and temperature-dependent fluid

Suppose Fr^2, $\mathrm{Fr}^2/\mathrm{Re}$ and Pe are of order of unity or greater, and the Brinkman number is of order $O(\varepsilon)$ or smaller. Then collecting the terms of order $O(1)$ in equations (66) and (67) and re-dimensionalizing lead to the non-isothermal Navier-Stokes equations:

$$\begin{cases} \rho_{\mathrm{ref}}\dot{\mathbf{v}} = -\nabla p + 2\mathrm{div}[\mu(p,\theta)\mathbf{D}] - \rho_{\mathrm{ref}} g\mathbf{k}, \\ \mathrm{div}\mathbf{v} = 0, \\ \rho_{\mathrm{ref}} c_p \dot{\theta} = \mathrm{div}[k(p,\theta)\nabla\theta]. \end{cases} \tag{76}$$

As pointed out earlier, for highly viscous fluids the Brinkman number might not be small [24] but of order of unity or greater. Hence, from (67) we deduce that equation $(76)_3$ has to be changed to

$$\rho_{\mathrm{ref}} c_p \dot{\theta} = \mathrm{div}[k(p,\theta)\nabla\theta] + 2\mu(p,\theta)\|\mathbf{D}\|^2. \tag{77}$$

5 Concluding remarks

In this paper we have derived some approximations for isochoric flows of fluids whose material parameters depend on both the temperature and the pressure. A crucial point in the mathematical derivation of such approximations is the assumption (3) according to which the deformation gradient is a function of the temperature and pressure. Contrary to the assumption (1), that is widely used in the literature to provide a mathematical justification for the Oberbeck-Boussinesq approximation (see [21, 22], for instance), according to which the fluid can sustain isochoric motions only in isothermal processes, our assumption does not have the attendant drawbacks of vanishing specific heat at constant volume, of thermodynamic instability and imaginary speed of sound in the fluid (see eqs. (36) and (45)).

If the deformation gradient is supposed to be a function only of the temperature the differential of the density is given by

$$\mathrm{d}\rho = -\rho\alpha\mathrm{d}\theta, \tag{78}$$

with $\alpha = -\dfrac{1}{\rho}\dfrac{\mathrm{d}\rho}{\mathrm{d}\theta}$. Consequently, the differential of the specific volume becomes

$$\mathrm{d}v = \frac{\alpha}{\rho}\mathrm{d}\theta \tag{79}$$

from which it is clear that isochoric motions are possible only in isothermal processes. As a consequence of this deduction, the specific heat at constant volume of a fluid satisfying assumption (1) is zero. In fact, from (79) and by following the same arguments as in sections 2 and 3 we get

$$\mathrm{d}e = -\frac{\alpha\theta}{\rho}\mathrm{d}p + \left(\frac{\rho c_p}{\alpha} - p\right)\mathrm{d}v \tag{80}$$

from which we deduce that the specific internal energy does not depend on the temperature in isochoric motions. In other words, $c_v = 0$. But, the specific heat at constant volume of fluids is not zero. For several common liquids $c_v \approx 0.9c_p (\neq 0)$, and for water at 15 °C $c_v = 0.997c_p (\neq 0)$ [2]. Consequently, assumption (1) is not suitable for common liquids and water.

On the other hand,

$$\mathrm{d}\eta = \frac{c_p}{\theta}\mathrm{d}\theta - \frac{\alpha}{\rho}\mathrm{d}p = \frac{1}{\theta}\mathrm{d}e + \frac{p}{\theta}\mathrm{d}v, \tag{81}$$

and, by combining (79) and (80),

$$\mathrm{d}\theta = \frac{\rho}{\alpha}\mathrm{d}v, \quad \mathrm{d}p = -\frac{\rho}{\alpha\theta}\mathrm{d}e + \frac{\rho}{\alpha^2\theta}(\rho c_p - \alpha p)\mathrm{d}v. \tag{82}$$

Then, from $(81)_2$ and (82), the hessian matrix of the specific entropy,

$$H_\eta(e, v) = \begin{pmatrix} 0 & -\dfrac{\rho}{\alpha\theta^2} \\ -\dfrac{\rho}{\alpha\theta^2} & \dfrac{\rho c_p - 2\alpha p}{\alpha^2\theta^2} \end{pmatrix}, \tag{83}$$

is not negative semi-definite as $\det H_\eta = -\dfrac{\rho^2}{\alpha^2\theta^4} < 0$. The specific entropy is then not concave and so assumption (1) leads also to thermodynamic instability (see also [13]).

Finally, from (78) and $(81)_1$ one gets the identity

$$\frac{1}{c^2} = \left(\frac{\partial\rho}{\partial p}\right)_\eta = -\frac{\alpha^2\theta}{c_p} < 0, \tag{84}$$

from which one deduces that the speed of sound in the fluid is imaginary, a physically unrealistic situation!

It must be acknowledged that, despite the physically unrealistic deductions described above, as long as the ratio between the Eckert number and the second Froude number is of the same order as the Carnot number or smaller, the approximations derived in [22], [30] and [24] under the assumption (1) (and valid whenever $\alpha_{\mathrm{ref}}(\theta_M - \theta_m)$ is small) coincide, respectively, with the Oberbeck-Boussinesq, Stokes-Oberbeck-Boussinesq and Navier-Stokes with frictional heating approximations derived in section 4. In virtue of this remark, the results concerning the onset of convection and the viscous dissipation in fluids with pressure- and temperature-dependent viscosity found in [23, 24, 30], where the approximations are derived under the assumption (1), hold true.

Data accessibility

Our paper has no data.

Competing intersts

We have no competing interests.

Authors' contributions

KRR partecipated in the design of the study and wrote some parts of the manuscript. GS coordinated the study and helped draft the manuscript. LV carried out the perturbation analysis, derived the approximations and helped draft the manuscript. All authors gave final approval for publication.

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